Exercise 1

a) x : susceptible people (fraction)

y : infected people

People not infected and not susceptible is...? Recovered!

→ $(1 - x - y)$ is the fraction of recovered people

→ α is the rate at which recovered become susceptible again (to re-infection)

b)

→ $\frac{dx}{dt} = 0 \qquad y = \frac{-\alpha + \alpha x}{-\alpha - x\beta}$

$$\alpha - \alpha x - \alpha y - xy\beta = 0$$

$$-\alpha y - xy\beta = -\alpha + \alpha x$$

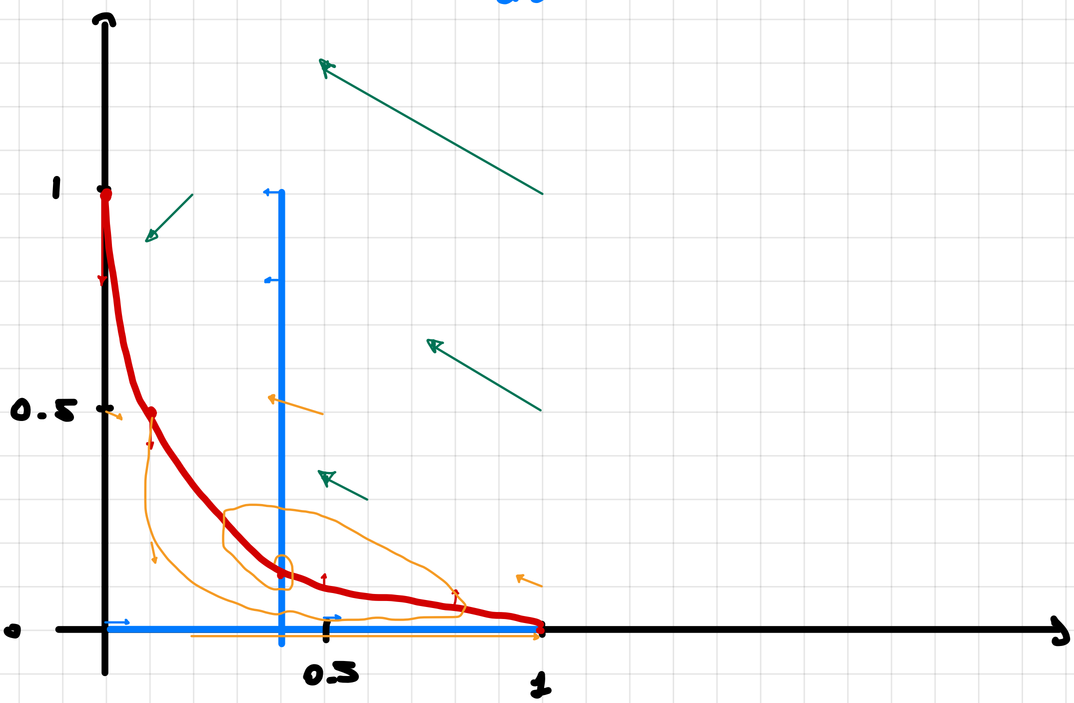
$$y(-\alpha - x\beta) = -\alpha + \alpha x$$

$\rightarrow \quad \frac{dy}{dt} = 0 \qquad y = 0 \quad \text{or} \quad x = \frac{1}{R_0}$

$$\left(x - \frac{1}{R_0}\right) y \beta = 0$$

$$y = \frac{-\alpha + \alpha x}{-\alpha - x\beta}$$

c) $\frac{dx}{dt} = 0 \qquad \frac{dy}{dt} = 0$



d) $(0, 0.5) \rightarrow (0.05, -0.2)$

$(0.1, 0.2) \rightarrow (0.05, -0.06)$

$(0.5, 0.5) \rightarrow (-0.25, 0.05)$

$(0.9, 0.1) \rightarrow (-0.09, 0.05)$

Where's e ?!

8) For $(x, y) = (0.5, 0.5)$ we have

$$\left(\frac{dx}{dt}, \frac{dy}{dt}\right) = (-0.25, 0.05)$$

We can use $\Delta t = 0.5$ which gives

$$(\Delta x, \Delta y) = (-0.125, 0.025)$$

See plot!

9) $\frac{dy}{dt}$ at $(0, 1) = -0.4$ $\quad \Delta y = -0.2$

$\frac{dy}{dt}$ at $(0.1, 0.5) = -0.15$ $\quad \Delta y = -0.075$

$\frac{dy}{dt}$ at $(0.5, 0.1) = 0.01$ $\quad \Delta y = 0.005$

$\frac{dy}{dt}$ at $(0.8, 0.05) = 0.02$ $\quad \Delta y = 0.01$

$$\frac{dx}{dt} \text{ at } (0,0) = 0.1 \qquad \Delta x = 0.05$$

$$\frac{dx}{dt} \text{ at } (0, 0.5) = 0.05 \qquad \Delta x = 0.025$$

$$\frac{dx}{dt} \text{ at } (0.4, 1) = -0.44 \qquad \Delta x = -0.22$$

$$\frac{dx}{dt} \text{ at } (0.4, 0.8) = -0.38 \qquad \Delta x = -0.19$$

(See plot!)

b)

(See plot!)

(x, y)	$\left(\frac{dx}{dt}, \frac{dy}{dt}\right)$	$(\Delta x, \Delta y)$
$(1, 1)$	$(-1.1, 0.6)$	$(-0.55, 0.3)$
$(1, 0.5)$	$(-0.55, 0.3)$	$(-0.27, 0.15)$
$(0.7, 0.3)$	$(-0.21, 0.09)$	$(-0.1, 0.045)$
$(0.2, 1)$	$(-0.22, -0.2)$	$(-0.1, -0.1)$

i) See plot!

j) See plot!

k)

→ Recalling that $dx/dt = 0$ at

$$y = \frac{-\alpha + \alpha x}{-\alpha - x\beta}$$

When $\alpha \to 0$ we have

$$y \sim \frac{a x}{-x\beta} \simeq -\frac{\alpha}{\beta}$$

The x nullcline flattens slighty above the y nullcline.

→ Flow arrows get more negative x components. People recover less quickly?

UNSURE!

→ At the limit $\alpha \to 0$ x nullcline flattens on y nullcline. Fixed points everywhere!

Exercise 2

a)

$$\rightarrow h_i(t) = \sum_j w_{ij} S_j(t) =$$

$$= \sum_j \frac{J}{(1-a^2)N} \sum_\mu \left(\xi_i^\mu\right)\left(\xi_j^\mu - a\right) S_j(t)$$

$$= \sum_\mu \frac{J}{(1-a^2)N} \xi_i^\mu \sum_j \left(\xi_j^\mu - a\right) S_j(t)$$

$$= J \sum_\mu \xi_i^\mu \, m^\mu(t)$$

$$\ldots = 0.5 \left[1 + g\left(J \sum_\mu \xi_i^\mu m^\mu(t)\right)\right]$$

$$\rightarrow \frac{1}{N(1-a^2)} \sum_j \left(\xi_j^\mu - a\right) \xi_j^\mu =$$

$$= \frac{1}{N(1-a^2)} \sum_j \left(1 - a\,\xi_j^\mu\right) =$$

$$= \frac{1}{N(1-a^2)} \left(N - a \sum \xi_j^\mu\right) =$$

$$\sum \xi_j^\mu = (a+1)\,N/2 - \left(N - (a+1)\frac{N}{2}\right)$$

$+1$ s $\qquad -1$ s

$$= (a+1)N - N = aN$$

$$= \frac{1}{N(1-a^2)} \left(N - a^2 N\right) = 1$$

b) Overlap is a dot-product-like measure of how close the current activation is to each of the memorized patterns.

c)

$$P[S_i(1)=1] = 0.5\left[1+g\left(J\sum_\mu \xi_i^\mu m^\mu(t)\right)\right] =$$

$$= 0.5\left[1+g\left(J\xi_i^4 M\right)\right] =$$

$$= 0.5\left[1+g\left(-JM\right)\right]$$

For $J=1,\ M=0.2$

$$P[S_i(1)=1] = 0.5\left[1+g(-0.2)\right] =$$

$$0.5\left[1-0.8\right] = 0.1$$

d) With $J=1,\ M=0.2$ we have

$$P[S_i(1)] = \xi_i^4 = 0.9$$

$$m^4(1) = \frac{1}{N(1-a^2)}\left(\sum_{j \in \text{match}} (\xi_j^4 - a)\,\xi_j^4 - \sum_{j \in \text{mismatch}} (\xi_j^4 - a)\,\xi_j^4\right)$$

$$= \frac{|\text{match}|}{N} - \frac{|\text{mismatch}|}{N}$$

$$\langle m^4(1)\rangle = 0.9 - 0.1 = 0.8$$

$$\langle m^1(1)\rangle = 0$$ Matches balance out, mismatches balance out

e) Yes. The number of matches / mismatches between $S(1)$ and ξ^4 is binomially distributed. As $N \to \infty$, standard deviation grows with $\sqrt{N}$ only, and fraction of matches (which determines m) converges to having 0 standard deviation.

f)

→ We have

$$P[S_i(t+1) = \xi_i^4] = 0.5\left[1 + g\left(J\, m^4(t)\right)\right]$$

So we have

$$m^4(t+1) = 2\left(0.5\left(1 + g\left(\textstyle\sum m^4(t)\right)\right)\right) - 1$$

$$= 1 + g\left(m^4(t)\right) - 1 = g\left(m^4(t)\right)$$

Which we can solve for $m = m^4(\infty)$ by

$$m = g(m)$$

$$m = 1$$

Final overlap with pattern 4 is 1.
Final overlap with pattern 0 is 0.

→ $m^4(0) = 0.2$

$m^4(1) = 0.8$

$m^4(2) = 1$

Takes two timesteps to converge! Cool!

g)

$$W_{ij} = \frac{1}{(1-a^2)N} \sum_{\mu} \left(\xi_i^{m+1}\right)\left(\xi_j^{\mu} - a\right)$$

$$h_i(t) = \sum_{\mu} \xi_i^{\mu+1} m^{\mu}(t)$$

So essentially I rotate pattern at every timestep!

$$m^{i+1}(t+1) = g\left(m^i(t)\right)$$

And with $m^4(0) = 0.5$ we get

$m^1(1) = 0$ $\quad$ $m^5(1) = 1$

$m^4(1) = 0$ $\quad$ $m^6(1) = 0$

Exercise 3

a) → h_0 is the neuron's resting potential

→ J represents how tightly neurons are connected to each other

b) ?

FIND OUT!

c) We want $\frac{dh}{dt} = 0$

$$h = g(h)$$

1 st branch: $h = 0.2$

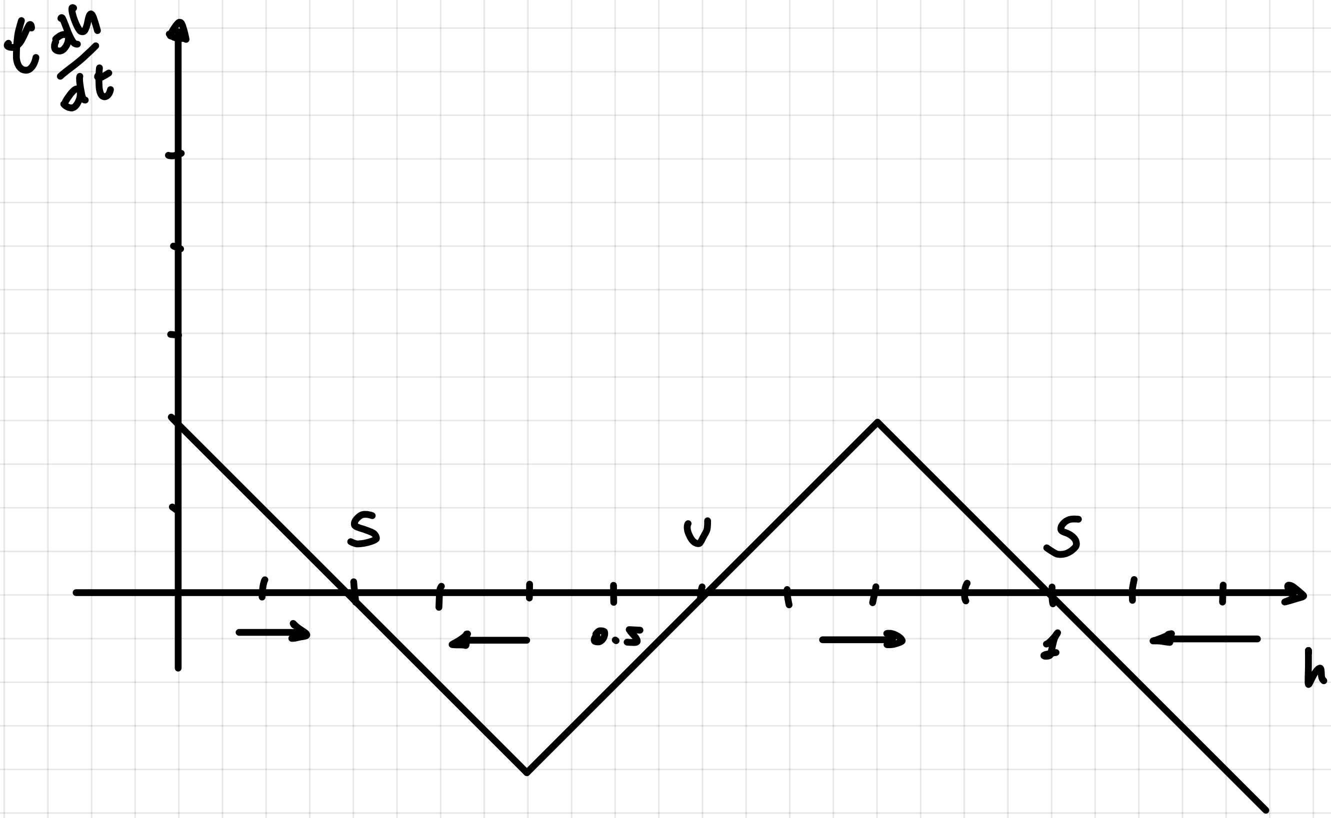
3 rd branch: $h = 1$

2 nd branch: $h = 2h - 0.6$

$h = 0.6$

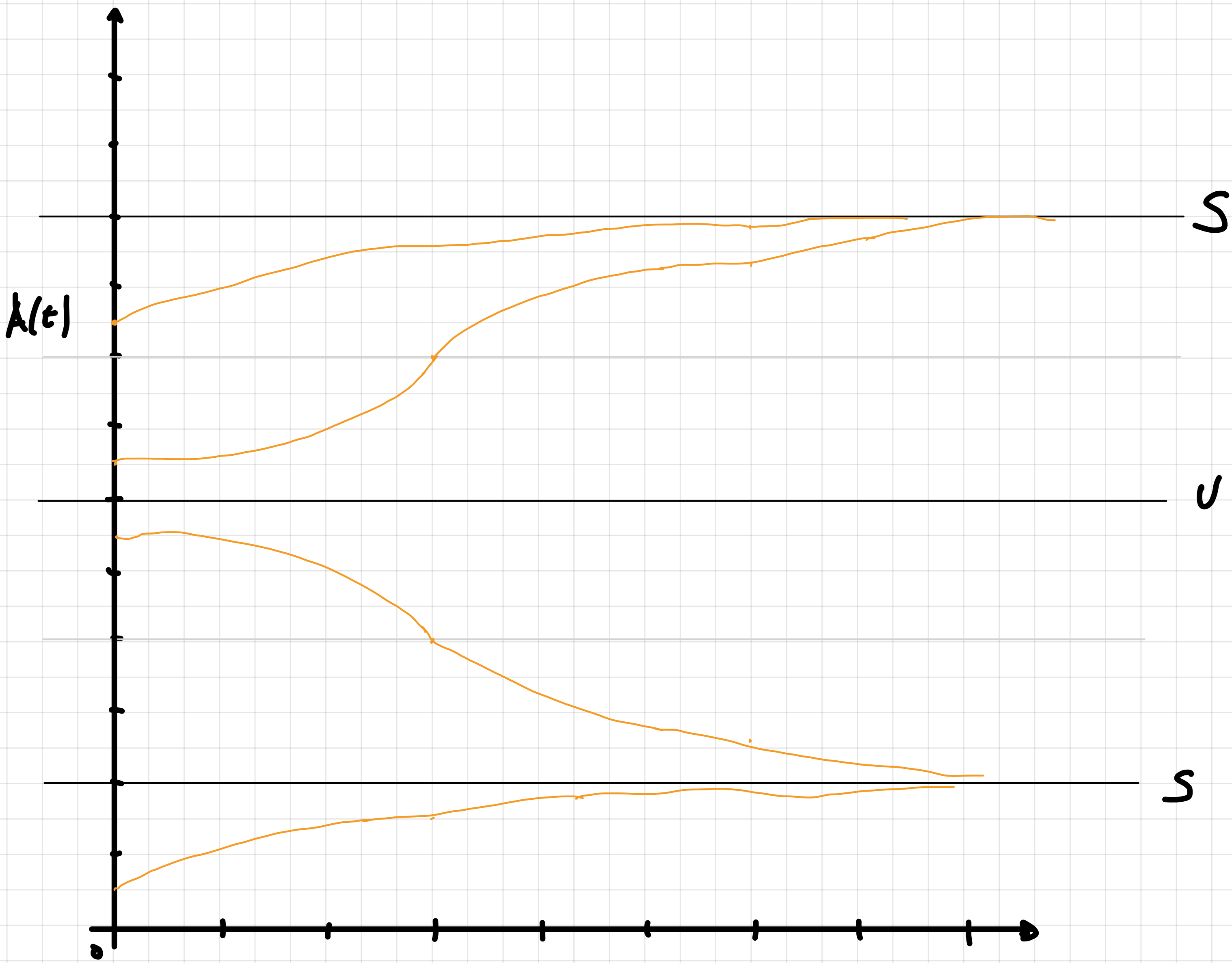
d) $\tau \dfrac{dh}{dt} = g(h) - h$

$$= \begin{cases} 0.2 - h & h < 0.4 \\ h - 0.6 & h \in [0.4, 0.8] \\ 1 - h & h > 0.8 \end{cases}$$



e) Everything converges exponentially to / from the relevant fixed point by $e^{\pm t/t}$

So we need to divide / multiply the distance by 3 every 3 timesteps



Exercise 4

$$\alpha(t) = \sigma(t) + \sigma(t - t_0)$$

$$\sigma(t) = \begin{cases} b\, e^{-t/t_m} & \text{iff } t > 0 \\ 0 & \text{otherwise} \end{cases}$$

d) Each spike doublet elicits in total

$$\int_0^\infty \alpha(t)\, dt = 2 \int_0^\infty \sigma(t)\, dt =$$

$$= 2b \int_0^\infty e^{-t/t_m}\, dt = 2 b\, t_m$$

CORRECT! NUMERICALLY VERIFIED!

So the expected voltage is

$$\langle v \rangle = \underbrace{2 b\, t_m}_{\text{Single doublet}}\; \nu_0\; K$$

ν_0: Doublets per second per neuron

K: Number of neurons

b1)

$$\langle v \rangle = 2 \cdot 2\,mV \cdot 5\,ms \cdot 0.5\,Hz \cdot 1000 =$$

$$= 2 \cdot 2\,mV \cdot 5\,\cancel{ms} \cdot 5 \cdot 10^{-4} / \cancel{ms} \cdot 1000 =$$

$$= (2 \cdot 2 \cdot 5 \cdot 0.5)\,mV = 10\,mV$$

b2) No, the voltage will hover around a half / a third of the threshold. If K was smaller it could fire by fluctuation!

c) Mean input potential would double (the time integral of each spike would halve, but we would have four times as many).